

THE EXISTENCE AND CONCENTRATION OF POSITIVE GROUND STATE SOLUTIONS FOR A CLASS OF FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH STEEP POTENTIAL WELL

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ABSTRACT. The present study is concerned with the following fractional Schrödinger-Poisson system with steep potential well:

$$\begin{cases} (-\Delta)^s u + \lambda V(x)u + K(x)\phi u = f(u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $s, t \in (0, 1)$ with $4s + 2t > 3$, and $\lambda > 0$ is a parameter. Under certain assumptions on $V(x)$, $K(x)$ and $f(u)$ behaving like $|u|^{q-2}u$ with $2 < q < 2_s^* = \frac{6}{3-2s}$, the existence of positive ground state solutions and concentration results are obtained via some new analytical skills and Nehari-Pohožaev identity. In particular, the monotonicity assumption on the nonlinearity is not necessary.

1. INTRODUCTION AND MAIN RESULTS

In the present paper, we are concerned with the existence and concentration of positive ground state solutions for the following fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta)^s u + \lambda V(x)u + K(x)\phi u = f(u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $s, t \in (0, 1)$, $4s + 2t > 3$ and the parameter $\lambda > 0$. On the potential $V(x)$, we need to make the following assumptions:

(V₁) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ with $V(x) \geq 0$ on \mathbb{R}^3 ;

(V₂) there exists $c > 0$ such that the set $\{V < c\} \triangleq \{x \in \mathbb{R}^3 : V(x) < c\}$ has positive finite Lebesgue measure;

(V₃) $\Omega = \text{int}V^{-1}(0)$ is nonempty and has smooth boundary with $\overline{\Omega} = V^{-1}(0)$, where $V^{-1}(0) \triangleq \{x \in \mathbb{R}^3 : V(x) = 0\}$.

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In their celebrated paper, T. Bartsch and Z. Wang [8] firstly proposed the above hypotheses to study a nonlinear Schrödinger equation. The potential $\lambda V(x)$ with assumptions $(V_1) - (V_3)$ usually are called by the steep potential well.

Let us recall the history of the study for Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \quad (1.2)$$

Due to the real physical meaning, the system (1.2) has been studied extensively by many scholars in the last several decades. Benci and Fortunato [10] introduced the system like (1.2) to describe solitary waves for nonlinear Schrödinger type equations and look for the existence of standing waves interacting with an unknown electrostatic field. We refer the readers to [10, 11] and the references therein to get a more physical background of the system (1.2). Nearly Y. Jiang and H. Zhou [24] firstly applied the steep potential well to the Schrödinger-Poisson system and proved the existence of nontrivial solutions and ground state solutions. Subsequently by using the linking theorem [31, 43], L. Zhao, H. Liu and F. Zhao [47] studied the existence and concentration of nontrivial solutions for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.3)$$

under the conditions

(\widetilde{V}_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded from below;

and $(V_2) - (V_3)$ with some suitable assumptions on $K(x)$ for $4 \leq p < 6$. It is worth mentioning that they specially established the existence and concentration of nontrivial solutions to (1.3) by L. Jeanjean's monotonicity trick [22] under the conditions $(V_1) - (V_3)$, $K(x) \geq 0$ for $x \in \mathbb{R}^3$ with $K(x) \in L_{\text{loc}}^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and

(\widetilde{V}_4) $V(x)$ is weakly differentiable such that $(x, \nabla V) \in L^{p_1}(\mathbb{R}^3)$ for some $p_1 \in [\frac{3}{2}, \infty]$, and

$$2V(x) + (x, \nabla V) \geq 0, \text{ for a.e. } x \in \mathbb{R}^3,$$

where (\cdot, \cdot) is the usual inner product in \mathbb{R}^3 .

(\widetilde{K}) $K(x)$ is weakly differentiable such that $(x, \nabla K) \in L^{p_2}(\mathbb{R}^3)$ for some $p_2 \in [2, \infty]$, and

$$\frac{2(p-3)}{p}K(x) + (x, \nabla K) \geq 0, \text{ for a.e. } x \in \mathbb{R}^3.$$

Replaced $|u|^{p-2}u$ by $a(x)f(u)$ in (1.3), Du *et.al* [17] proved the existence and asymptotic behavior of solutions under conditions $(V_1) - (V_3)$ or $(\widetilde{V}_1) - (V_2) - (V_3)$ and some suitable assumptions $a(x)$ and $K(x)$, where $\lim_{t \rightarrow \infty} f(t)/t = l \in (0, +\infty)$. There are many interesting works about the existence of positive solutions, positive ground states, multiple solutions, sign-changing solutions and semiclassical states to (1.2), see e.g. [1, 2, 3, 6, 7, 20, 21, 32, 33, 35, 36, 39, 45, 48] and their references therein.

The nonlinear fractional Schrödinger-Poisson systems (1.1) come from the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N \quad (1.4)$$

used to study the standing wave solutions $\psi(x, t) = u(x)e^{-i\omega t}$ for the equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hbar^2 (-\Delta)^\alpha \psi + W(x)\psi - f(x, \psi) \quad x \in \mathbb{R}^N,$$

where \hbar is the Planck's constant, $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is an external potential and f a suitable nonlinearity. Since the fractional Schrödinger equation appears in problems involving nonlinear optics, plasma physics and condensed matter physics, it is one of the main objects of the fractional quantum mechanics. The equation (1.4) has been firstly proposed by Laskin [25, 26] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In their celebrated paper, Caffarelli-Silvestre [15] transform the nonlocal operator $(-\Delta)^\alpha$ to a Dirichlet-Neumann boundary value problem for a certain elliptic problem with local differential operators defined on the upper half space. This technique of Caffarelli-Silvestre is a valid tool to deal with the equations involving fractional operators in the respects of regularity and variational methods, please see [2, 20] and their references for example. When the conditions $(V_1) - (V_3)$ are satisfied, L. Yang and Z. Liu [44] proved the multiplicity and concentration of solutions for the following fractional Schrödinger equation

$$(-\Delta)^s u + \lambda V(x)u = f(x, u) + \alpha(x)|u|^{v-2}u, \quad x \in \mathbb{R}^N,$$

involving a k -order asymptotically linear term $f(x, u)$, where $s \in (0, 1)$, $2s < N$, $1 \leq k < 2_s^* - 1 = \frac{N+2s}{N-2s}$ and $\alpha \in L^{\frac{N}{2-s}}(\mathbb{R}^N)$ with $1 < v < 2$. Please see [4, 5, 13, 18, 19] and their references for some other related results on fractional Schrödinger equation.

However similar results on the fractional Schrödinger-Poisson systems are not as rich as the Schrödinger-Poisson system (1.2), especially there are very few results on the existence and concentration results with steep potential well. Very recently, K. Teng and R. Agarwal [41] considered the semiclassical case for the following fractional Schrödinger-Poisson system

$$\begin{cases} \epsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = K(x)f(u) + Q(x)|u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ \epsilon^{2t}(-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

under some appropriate conditions on $K(x)$, $Q(x)$ and $f \in C^1(\mathbb{R}^3)$ behaving like $|u|^{p-2}u$ with $4 < p < 2_s^* = \frac{6}{3-2s}$, where the existence and concentration of positive ground state solutions were obtained. Other interesting results on fractional Schrödinger-Poisson system can be found in [28, 29, 37, 40, 42, 46] and their references.

Motivated by all the works just described above, particularly by [47], we prefer to investigate the existence and concentration results for (1.1) with steep potential well and more general nonlinearity. Since we are interested in positive solutions, without loss of generality, we assume that $f \in C^0(\mathbb{R}, \mathbb{R})$ vanishes in $(-\infty, 0)$ and satisfies the following conditions:

- (f₁) $f \in C^0(\mathbb{R}, \mathbb{R}^+)$ and $f(z) = o(z)$ as $z \rightarrow 0$, where $\mathbb{R}^+ = [0, +\infty)$;
- (f₂) $|f(z)| \leq C_0(1 + |z|^{q-1})$ for some constants $C_0 > 0$ and $2 < q < 2_s^* = \frac{6}{3-2s}$;
- (f₃) there exist a constant $\gamma > \frac{4s+2t}{s+t}$ such that $zf(z) - \gamma F(z) \geq 0$, where $F(z) = \int_0^z f(s)ds$.

Our main results are as follows:

Theorem 1.1. *Let $s, t \in (0, 1)$ satisfy $4s + 2t > 3$, and assume that $(V_1) - (V_3)$, $(f_1) - (f_3)$, $K(x) \geq 0$ for all $x \in \mathbb{R}^3$ with $K(x) \in L^\infty(\mathbb{R}^3) \cap L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ with $s \geq t$. In addition, we assume the following conditions:*

- (V₄) $V(x)$ is weakly differentiable and $(x, \nabla V) \in L^\infty(\mathbb{R}^3) \cup L^{\frac{3}{2s}}(\mathbb{R}^3)$ verifies the following inequality:

$$(s+t)(\gamma-2)V(x) + (x, \nabla V) \geq 0,$$

where (\cdot, \cdot) is the usual inner product in \mathbb{R}^3 .

- (K) $K(x)$ is weakly differentiable and $(x, \nabla K) \in L^\infty(\mathbb{R}^3) \cup L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ satisfies the following inequality:

$$[(s+t)\gamma - (4s+2t)]K(x) + 2(x, \nabla K) \geq 0.$$

Then there exists $\Lambda > 0$ such that the system (1.1) admits at least one positive ground state solution for all $\lambda > \Lambda$.

Remark 1.2. *There are some remarks on Theorem 1.1 as follows:*

- (1) *The hypothesis $K(x) \in L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ with $s \geq t$ is unnecessary if we restrict the work spaces to radially symmetric spaces, such as $H_r^s(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) : u(x) = u(|x|)\}$. In other words if the work spaces are radially symmetric, we may have $\gamma \leq 3$ which is an interesting phenomenon, where the positive constant γ comes from (f₃).*
- (2) *Compared with the conditions $(\widetilde{V}_4) - (\widetilde{K})$ in [47] and $(V_4) - (K)$ in our paper, we have to make a carefully analysis to the fractional Schrödinger-Poisson system involving a more general nonlinearity. On the other hand, we always assume $q \in (2, 2_s^*)$ in (f₂), hence the assumptions $(V_4) - (K)$ are never redundant.*
- (3) *It should pointed out here that the above nonlinearity assumptions $(f_1) - (f_3)$ mainly were motivated by J. Sun and S. Ma [38]. Compared with [38], some appropriate modifications were made to adapt the fractional Schrödinger-Poisson system.*
- (4) *A typical example of the nonlinearity verifying the assumptions $(f_1) - (f_3)$ is given by $f(z) = |z|^{\gamma-2}z$ with $\gamma > \frac{4s+2t}{s+t}$.*

Remark 1.3. *Recently, K. Teng [40] and Shen-Yao [37] have considered the existence of ground state solutions to the following fractional Schrödinger-Poisson system:*

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = |u|^{p-2}u + \mu|u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

with $\mu \geq 0$ and $2 < p < 2_s^*$ under suitable assumptions of $V(x)$. The two papers above were required to meet condition $2s + 2t > 3$, which is more restricted than the condition $4s + 2t > 3$ in this paper if $f(u)$ behaves like $|u|^{q-2}u$ with $2 < q < 2_s^*$. In fact, we remark that by the techniques here, the condition $2s + 2t > 3$ can be improved to the inequality $4s + 2t > 3$.

Inspired by the results in [9, 17, 24, 44, 47], we get the following concentration result:

Theorem 1.4. *Let $(u_\lambda, \phi_{u_\lambda})$ be the nontrivial solutions obtained in Theorem 1.1, then $u_\lambda \rightarrow u_0$ in $H^s(\mathbb{R}^3)$ (see Section 2 below) and $\phi_{u_\lambda} \rightarrow \phi_{u_0}$ in $D^{t,2}(\mathbb{R}^3)$ (see Section 2 below) as $\lambda \rightarrow +\infty$, where $u_0 \in H_0^s(\Omega)$ is a nontrivial solution to*

$$\begin{cases} (-\Delta)^s u + c_t \left((K(x)u^2) * \frac{1}{|x|^{3-2t}} \right) K(x)\phi u = f(u), & x \in \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

Note that $c_t > 0$ is a constant from (2.8) below.

Now we give our main ideas for the proofs of Theorem 1.1 and 1.4. It is not simple to verify that I_λ (see Section 2) possesses a Mountain-pass geometry in the usual way because the Ambrosetti-Rabinowitz type condition ((AR) in short):

(AR) There exists $\eta > 4$ such that $0 < \eta F(t) \leq f(t)t$ for all $t \neq 0$

or 4-superlinear at infinity in the sense that

$$(F) \quad \lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^4} = +\infty.$$

does not always hold. Furthermore, even if a (PS) sequence has been obtained, it is difficult to prove its boundedness since the nonlinearity $f(u)$ behaving like $|u|^{q-2}u$ with $2 < q < 2_s^*$ results in neither the weaker condition $(AR)_4$ ($\eta = 4$ in (AR)) nor the condition

(M) The map $t \rightarrow \frac{f(t)}{t^3}$ is positive for $t \neq 0$, strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$.

works yet. To overcome this difficulties, motivated by [48], we use an indirect approach (see Proposition 2.4) developed by L. Jeanjean [23] to get a bounded (PS) sequence. Though a bounded (PS) sequence can be constructed, another difficulty on the lack of compactness of the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ with $2 \leq r \leq 2_s^*$ occurs and the (PS) condition seems to be hard to verify because we do not assume the potential $V(x)$ and the weight function $K(x)$ to be radially symmetric. To solve it, we assume $K(x) \in L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ with $s \geq t$ to recover the compactness and then to prove the (PS) condition. So far, we can prove the Theorem 1.1 and 1.4 step by step.

The paper is organized as follows. In Section 2, the function spaces will be introduced and then we provide several lemmas, which are crucial in proving our main results. In Section 3, the proof of Theorem 1.1 is obtained. The concentration result of Theorem 1.4 will be proved in Section 4.

Notations. Throughout this paper we shall denote by C and C_i ($i = 1, 2, \dots$) for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem. $L^p(\mathbb{R}^3)$ ($1 \leq p \leq +\infty$) is the usual Lebesgue space with the standard norm $\|u\|_p$. We use “ \rightarrow ” and “ \rightharpoonup ” to denote the strong and weak convergence in the related function space, respectively. The symbol “ \hookrightarrow ” means a function space is continuously imbedding into another function space. The Lebesgue measure of a Lebesgue measurable set E in \mathbb{R}^3 is $|E|$. For any $\rho > 0$ and any $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius ρ centered at x , that is, $B_\rho(x) := \{y \in \mathbb{R}^3 : |y - x| < \rho\}$.

Let $(X, \|\cdot\|)$ be a Banach space with its dual space $(X^{-1}, \|\cdot\|_*)$, and Φ be its functional on X . The Palais-Smale sequence at level $c \in \mathbb{R}$ ($(PS)_c$ sequence in short) corresponding to Φ assumes that $\Phi(x_n) \rightarrow c$ and $\Phi'(x_n) \rightarrow 0$ as $n \rightarrow \infty$, where $\{x_n\} \subset X$. If for any $(PS)_c$ sequence $\{x_n\}$ in X , there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow x_0$ in X for some $x_0 \in X$, then we say that the functional Φ satisfies the so called $(PS)_c$ condition.

2. VARIATIONAL SETTINGS AND PRELIMINARIES

In this section, we first bring in some necessary variational settings for system (1.1) and the complete introduction to the fractional Sobolev spaces can be found in [30]. Recalling that the fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ is defined for any $p \in [1, +\infty)$ and $\alpha \in (0, 1)$ as follows

$$W^{\alpha,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\alpha p}} dx dy < +\infty \right\}$$

equipped with the natural norm

$$\|u\|_{W^{\alpha,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\alpha p}} dx dy + \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}.$$

In particular, if $p = 2$, the fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^N)$ is simply denoted by $H^\alpha(\mathbb{R}^N)$. As we all know, the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ can be also described by the Fourier transform, that is,

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 d\xi < +\infty \right\},$$

where \widehat{u} denotes the usual Fourier transform of u . When we take the definition of the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ by the Fourier transform, the inner product and the norm for $H^\alpha(\mathbb{R}^N)$ are defined as

$$(u, v)_0 = \int_{\mathbb{R}^N} |\xi|^{2\alpha} \widehat{u}(\xi) \widehat{v}(\xi) + \widehat{u}(\xi) \widehat{v}(\xi) d\xi$$

and

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 + |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Following from Plancherel's theorem, one has $|u|_2 = |\widehat{u}|_2$ and $|(-\Delta)^{\frac{\alpha}{2}} u|_2 = \|\xi\|^\alpha \widehat{u}\|_2$. Hence

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 + |u|^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in H^\alpha(\mathbb{R}^N). \quad (2.1)$$

As a consequence of [30, Proposition 3.4 and Proposition 3.6], one has

$$|(-\Delta)^{\frac{\alpha}{2}} u|_2 = \left(\int_{\mathbb{R}^N} |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \left(\frac{1}{C_N(\alpha)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy \right)^{\frac{1}{2}},$$

which reveals that the norm given by (2.1) makes sense for the fractional Sobolev space. Meanwhile the homogeneous fractional Sobolev space $D^{\alpha,2}(\mathbb{R}^N)$ is defined by

$$D^{\alpha,2}(\mathbb{R}^N) = \left\{ u \in L^{2_\alpha^*}(\mathbb{R}^N) : |\xi|^\alpha \widehat{u}(\xi) \in L^{2_\alpha^*}(\mathbb{R}^N) \right\} \text{ with } 2_\alpha^* = \frac{2N}{N-2\alpha} \text{ and } N \geq 3.$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{\alpha,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^N} |\xi|^{2\alpha} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

The following fractional Sobolev embedding theorems are necessary.

Lemma 2.1. (see [27]) *For any $\alpha \in (0, \frac{N}{2})$, $H^\alpha(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for $r \in [2, 2_\alpha^*]$ and compactly embedded into $L_{loc}^r(\mathbb{R}^N)$ for $r \in [1, 2_\alpha^*)$.*

As a direct consequence of Lemma 2.1, there are constants $C_r > 0$ such that

$$\|u\|_{H^\alpha(\mathbb{R}^N)} \leq C_r |u|_r, \quad \forall u \in H^\alpha(\mathbb{R}^N) \text{ and } 2 \leq r \leq 2_\alpha^*. \quad (2.2)$$

Also there exists a best constant $S_\alpha > 0$ (see [16]) such that

$$S_\alpha = \inf_{u \in D^{\alpha,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2_\alpha^*} dx \right)^{\frac{2}{2_\alpha^*}}}. \quad (2.3)$$

In this paper, for $s, t \in (0, 1)$ we restrict the work spaces in dimension $N = 3$ and let

$$E \triangleq \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 dx < +\infty \right\}$$

be endowed with the inner product and the norm

$$(u, v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x) uv dx, \quad \|u\| = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + V(x) u^2 dx \right)^{\frac{1}{2}}$$

for any $u, v \in E$. By using the assumptions $(V_1) - (V_2)$ and (2.3), one has

$$\begin{aligned} \int_{\mathbb{R}^3} u^2 dx &= \int_{\{V \geq c\}} u^2 dx + \int_{\{V < c\}} u^2 dx \\ &\leq \frac{1}{c} \int_{\{V \geq c\}} V(x) u^2 dx + |\{V < c\}|^{\frac{2_\alpha^*-2}{2s}} \left(\int_{\{V < c\}} |u|^{2_\alpha^*} dx \right)^{\frac{2}{2_\alpha^*}} \\ &\leq \max \left\{ \frac{1}{c}, |\{V < c\}|^{\frac{2_\alpha^*-2}{2s}} \right\} \|u\|^2 \end{aligned}$$

which implies that the imbedding $E \hookrightarrow H^s(\mathbb{R}^3)$ is continuous. Thus by (2.2) there exists $d_r > 0$ such that

$$|u|_r \leq d_r \|u\|, \quad \forall u \in E \text{ and } 2 \leq r \leq 2_s^*. \quad (2.4)$$

For any $\lambda > 0$, we let $E_\lambda \triangleq (E, \|\cdot\|_\lambda)$ and the inner product and norm are

$$(u, v)_\lambda = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + \lambda V(x) u v dx, \quad \|u\|_\lambda = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x) |u|^2 dx \right)^{\frac{1}{2}}.$$

Obviously, $\|u\| \leq \|u\|_\lambda$ if $\lambda \geq 1$. The following facts

$$\int_{\{V < c\}} |u|^2 dx \leq |\{V < c\}|^{\frac{2_s^*-2}{2_s^*}} |u|_{2_s^*}^2 \stackrel{(2.3)}{\leq} |\{V < c\}|^{\frac{2_s^*-2}{2_s^*}} S_s^{-1} \|u\|_\lambda^2$$

and

$$\int_{\{V \geq c\}} |u|^2 dx \leq \frac{1}{\lambda c} \int_{\{V \geq c\}} \lambda V(x) |u|^2 dx \leq \frac{1}{\lambda c} \int_{\mathbb{R}^3} \lambda V(x) |u|^2 dx \leq \frac{1}{\lambda c} \|u\|_\lambda^2$$

give us that for any $r \in [2, 2_s^*]$

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^r dx &\leq \left(\int_{\mathbb{R}^3} |u|^2 dx \right)^{\frac{2_s^*-r}{2_s^*-2}} \left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx \right)^{\frac{r-2}{2_s^*-2}} \\ &\stackrel{(2.3)}{\leq} \left(2 \max \left\{ S_s^{-1} |\{V < c\}|^{\frac{2_s^*-2}{2_s^*}}, \frac{1}{\lambda c} \right\} \|u\|_\lambda^2 \right)^{\frac{2_s^*-r}{2_s^*-2}} \left(S_s^{-\frac{2}{2_s^*}} \|u\|_\lambda^{2_s^*} \right)^{\frac{r-2}{2_s^*-2}}. \end{aligned}$$

Hence for any $r \in [2, 2_s^*]$, we have that

$$\int_{\mathbb{R}^3} |u|^r dx \leq \left(2 |\{V < c\}| \right)^{\frac{2_s^*-r}{2_s^*}} S_s^{-\frac{r}{2}} \|u\|_\lambda^r \text{ whenever } \lambda \geq c^{-1} |\{V < c\}|^{-\frac{2_s^*-2}{2_s^*}} S_s. \quad (2.5)$$

It is similar to the usual Schrödinger-Poisson system that the system (1.1) can reduce to be a single equation. Indeed, using the Hölder inequality, for every $u \in H^s(\mathbb{R}^3)$ and $v \in D^{t,2}(\mathbb{R}^3)$, one has

$$\begin{aligned} \int_{\mathbb{R}^3} K(x) u^2 v dx &\leq |K|_{\frac{6}{4s+2t-3}} \left(\int_{\mathbb{R}^3} |u|^{\frac{6}{3-2s}} dx \right)^{\frac{3-2s}{3}} \left(\int_{\mathbb{R}^3} |v|^{\frac{6}{3-2t}} dx \right)^{\frac{3-2t}{6}} \\ &\leq |K|_{\frac{6}{4s+2t-3}} S_s^{-1} S_t^{-\frac{1}{2}} \|u\|_{D^{t,2}(\mathbb{R}^3)}^2 \|v\|_{D^{t,2}(\mathbb{R}^3)} \leq C \|u\|^2 \|v\|_{D^{t,2}(\mathbb{R}^3)}, \end{aligned} \quad (2.6)$$

where we use the fact that $E \hookrightarrow H^s(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$. For any $u \in H^s(\mathbb{R}^3)$, one can use the Lax-Milgram theorem and then there exists a unique $\phi'_u \in D^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (-\Delta)^t \phi'_u v dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi'_u (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} K(x) u^2 v dx, \quad \forall v \in D^{t,2}(\mathbb{R}^3). \quad (2.7)$$

In other words, ϕ'_u satisfies the Poisson equation

$$(-\Delta)^t \phi'_u = K(x) u^2, \quad x \in \mathbb{R}^3$$

and we can write it an integral expression, that is,

$$\phi'_u(x) = c_t \int_{\mathbb{R}^3} \frac{K(x) u^2(y)}{|x-y|^{3-2t}} dy, \quad x \in \mathbb{R}^3, \quad (2.8)$$

which is called t -Riesz potential, where

$$c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3}{2} - 2t)}{\Gamma(t)}.$$

It follows from (2.8) that $\phi_u^t(x) \geq 0$ for all $x \in \mathbb{R}^3$. Taking $v = \phi_u^t$ in (2.6) and (2.7), we derive

$$\|\phi_u^t\|_{D^{t,2}(\mathbb{R}^3)} \leq C\|u\|^2. \quad (2.9)$$

Substituting (2.8) into (1.1), we can rewrite (1.1) in the following equivalent form

$$(-\Delta)^s u + \lambda V(x)u + K(x)(-\Delta)^t \phi_u^t u = f(u), \quad x \in \mathbb{R}^3. \quad (2.10)$$

The energy functional $I_\lambda : H^s(\mathbb{R}^3) \rightarrow \mathbb{R}$ associated to the problem (2.10) is given by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F(u) dx. \quad (2.11)$$

If we take $v = \phi_u^t$ in (2.6) and (2.7) again, we get

$$\int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx \leq C\|u\|^2 \|\phi_u^t\|_{D^{t,2}(\mathbb{R}^3)} \stackrel{(2.9)}{\leq} C\|u\|^4.$$

It is therefore that $I_\lambda(u)$ is well-defined and $I_\lambda \in C^1(E_\lambda, \mathbb{R})$ by (2.11) (see [43] for details), moreover its differential is

$$\langle I'_\lambda(u), v \rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} \lambda V(x) u v dx + \int_{\mathbb{R}^3} K(x) \phi_u^t u v dx - \int_{\mathbb{R}^3} f(u) v dx$$

for any $u, v \in E_\lambda$. It is clear that if u is a critical points of I_λ , then the pair (u, ϕ_u^t) is a solution of system (1.1).

Before giving the necessary lemmas for this paper, it is important to stress that the conditional assumptions in Theorem 1.1 and Theorem 1.4 are always true for simplicity. By simple calculations, we can deduce from (f_1) and (f_2) that

$$|f(u)| \leq \epsilon|u| + C_\epsilon|u|^{q-1} \quad \text{and} \quad |F(u)| \leq \epsilon u^2 + C_\epsilon|u|^q. \quad (2.12)$$

It follows from (f_1) and (f_2) that there exists a constant $C > 0$ such that

$$F(u) \geq C|u|^\gamma. \quad (2.13)$$

Lemma 2.2. Assume $K(x) \in L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ with $4s + 2t > 3$ and $s \geq t$, then the following properties are true:

(a) If $u \in H^s(\mathbb{R}^3)$ and we set $u_\theta(x) := \theta^{s+t} u(\theta x)$ for $\theta \in \mathbb{R}^+$, then

$$\int_{\mathbb{R}^3} \phi_{u_\theta}^t u_\theta^2 dx = \theta^{4s+2t-3} \int_{\mathbb{R}^3} \phi_u^t u^2 dx < +\infty.$$

(b) $\phi_{u(\cdot+y)}^t = \phi_u^t(x+y)$.

(c) If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^t \rightharpoonup \phi_u^t$ in $D^{t,2}(\mathbb{R}^3)$.

Proof. (a) Since $4s + 3t > 3$, then $u \in H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ and thus

$$\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq |\phi_u^t|_{2_t^*} |u|_{\frac{12}{3+2t}}^2 \stackrel{(2.3)}{\leq} S_t^{-\frac{1}{2}} \|\phi_u^t\|_{D^{t,2}(\mathbb{R}^3)} |u|_{\frac{12}{3+2t}}^2 \stackrel{(2.9)}{\leq} C \|u\|^2 |u|_{\frac{12}{3+2t}}^2 < +\infty.$$

By means of (2.8), one has

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_\theta}^t u_\theta^2 dx &= c_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_\theta^2(y) u_\theta^2(x)}{|x-y|^{3-2t}} dy dx \\ &= \theta^{4s+4t} \theta^{3-2t} \theta^{-6} c_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(y) u^2(x)}{|x-y|^{3-2t}} dy dx \\ &= \theta^{4s+2t-3} \int_{\mathbb{R}^3} \phi_u^t u^2 dx. \end{aligned}$$

(b) It is a direct consequence of (2.8).

(c) If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, by Lemma 2.1 and $\frac{3}{3-2s} \in (1, \frac{6}{3-2s})$, there exists a subsequence still denoted by itself such that $u_n \rightarrow u$ in $L_{\text{loc}}^{\frac{3}{3-2s}}(\mathbb{R}^3)$. Since $s \geq t$, then $\frac{6}{3-2t} \in (2, \frac{6}{3-2s}]$ and hence $|u_n + u|$ is uniformly bounded in $L^{\frac{6}{3-2t}}(\mathbb{R}^3)$. On the other hand for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, then $\varphi \in L^\infty(\mathbb{R}^3)$ and we have that

$$\left| \int_{\mathbb{R}^3} K(x) (u_n^2 - u^2) \varphi dx \right| \leq |\varphi|_\infty |K|_{\frac{6}{4s+2t-3}} |u_n + u|_{\frac{6}{3-2t}} \left(\int_{\text{supp } \varphi} |u_n - u|^{\frac{3}{3-2s}} dx \right)^{\frac{3-2s}{3}} \rightarrow 0,$$

where $\text{supp } \varphi$ denotes the support of φ . Since $C_0^\infty(\mathbb{R}^3)$ is dense in $H^s(\mathbb{R}^3)$, then the above formula shows that (c) is true. \square

The following lemma will play an vital role in recovering the compactness for the (PS) sequence, which is similar to the well-known Brézis-Lieb lemma [14].

Lemma 2.3. Assume $K(x) \in L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ with $4s + 2t > 3$ and $s \geq t$, if $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^3 , then we have that

$$\int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx \rightarrow 0, \quad (2.14)$$

and

$$\int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n \varphi dx - \int_{\mathbb{R}^3} K(x) \phi_u u \varphi dx \rightarrow 0 \quad (2.15)$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Proof. We point out here that the proof of the case $s = t = 1$ for this lemma can be found in [47], which can be viewed as a special one in our paper. Since $u \in H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{3-2s}}(\mathbb{R}^3)$, then one has

$$\int_{\mathbb{R}^3} |K|^{\frac{6}{3+2t}} |u|^{\frac{12}{3+2t}} dx \leq \left(\int_{\mathbb{R}^3} |K|^{\frac{6}{4s+2t-3}} dx \right)^{\frac{4s+2t-3}{3+2t}} \left(\int_{\mathbb{R}^3} |u|^{\frac{6}{3-2s}} dx \right)^{\frac{6-4s}{3+2t}} < +\infty$$

which implies that $Ku^2 \in L^{\frac{6}{3-2t}}(\mathbb{R}^3)$. By (c) of Lemma 2.2 and (2.3), one has $\phi_{u_n}^t \rightarrow \phi_u^t$ in $L^{\frac{6}{3-2t}}(\mathbb{R}^3)$ and thus

$$A_1 \triangleq \int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx \rightarrow 0.$$

On the other hand, $u_n \rightarrow u \in H^s(\mathbb{R}^3)$ gives that $|u_n - u| \rightarrow 0$ in $L^{\frac{6}{3-2s}}(\mathbb{R}^3)$ and then $|u_n - u|^{\frac{3}{s+t}} \rightarrow 0$ in $L^{\frac{2(s+t)}{3-2s}}(\mathbb{R}^3)$. Since $|K|^{\frac{3}{s+t}} \in L^{\frac{2(s+t)}{4s+2t-3}}(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} |K|^{\frac{3}{s+t}} |u_n - u|^{\frac{3}{s+t}} dx \rightarrow 0$$

which shows that

$$\begin{aligned} |A_2| &\triangleq \left| \int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx \right| \\ &\leq \left(\int_{\mathbb{R}^3} |K|^{\frac{3}{s+t}} |u_n - u|^{\frac{3}{s+t}} dx \right)^{\frac{s+t}{3}} \left(\int_{\mathbb{R}^3} |\phi_{u_n}^t|^{\frac{6}{3-2t}} dx \right)^{\frac{3-2t}{6}} \left(\int_{\mathbb{R}^3} |u_n + u|^{\frac{6}{3-2s}} dx \right)^{\frac{3-2s}{6}} \\ &\leq C \left(\int_{\mathbb{R}^3} |K|^{\frac{3}{s+t}} |u_n - u|^{\frac{3}{s+t}} dx \right)^{\frac{s+t}{3}} \rightarrow 0. \end{aligned}$$

Consequently, we have that

$$\int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx = A_1 + A_2 \rightarrow 0.$$

The proof of formula (2.15) is totally same as that of (2.14), so we omit it. \square

As described in Section 1, it is difficult for us to construct a bounded (PS) sequence because the conditions (AR), (M) and (F) do not hold. Thanks to the following well-known proposition, we can do it successfully.

Proposition 2.4. (See [22, Theorem 1.1 and Lemma 2.3]) *Let $(X, \|\cdot\|)$ be a Banach space and $T \subset \mathbb{R}^+$ be an interval, consider a family of C^1 functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in T,$$

with $B(u) \geq 0$ and either $A(u) \rightarrow +\infty$ or $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$. Assume that there are two points $v_1, v_2 \in X$ such that

$$c_\mu = \inf_{\gamma \in \Gamma} \sup_{\theta \in [0,1]} \Phi_\mu(\gamma(\theta)) > \max\{\Phi_\mu(v_1), \Phi_\mu(v_2)\}, \quad \forall \mu \in T,$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in T$, there is a sequence $\{u_n(\mu)\} \subset X$ such that

- (a) $\{u_n(\mu)\}$ is bounded in X ;
- (b) $\Phi_\mu(u_n(\mu)) \rightarrow c_\mu$ and $\Phi'_\mu(u_n(\mu)) \rightarrow 0$;
- (c) the map $\mu \rightarrow c_\mu$ is non-increasing and left continuous.

Letting $T = [\delta, 1]$, where $\delta \in (0, 1)$ is a positive constant. To apply Proposition 2.4, we will introduce a family of C^1 functionals on $X = E_\lambda$ with the form

$$I_{\lambda,\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x)|u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx - \mu \int_{\mathbb{R}^3} F(u) dx. \quad (2.16)$$

Then let $I_{\lambda,\mu}(u) = A(u) - \mu B(u)$, where

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + \lambda V(x)|u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx \rightarrow +\infty \text{ as } \|u\|_\lambda \rightarrow +\infty,$$

and

$$B(u) = \int_{\mathbb{R}^3} F(u) dx \geq 0.$$

It is clear that $I_{\lambda,\mu}$ is a well-defined C^1 functional on the space E_λ , and for all $u, v \in E_\lambda$, one has

$$\langle I'_{\lambda,\mu}(u), v \rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + \lambda V(x) u v dx + \int_{\mathbb{R}^3} K(x) \phi_u^t u v dx - \mu \int_{\mathbb{R}^3} f(u) v dx.$$

We now in a position to verify the Mountain-pass geometry for the functional $I_{\lambda,\mu}$.

Lemma 2.5. *The functional $I_{\lambda,\mu}$ possesses a Mountain-pass geometry, that is,*

- (a) *there exists $v \in E \setminus \{0\}$ independent of μ such that $I_{\lambda,\mu}(v) \leq 0$ for all $\mu \in [\delta, 1]$;*
- (b) *$c_{\lambda,\mu} \triangleq \inf_{\gamma \in \Gamma} \sup_{\theta \in [0,1]} I_{\lambda,\mu}(\gamma(\theta)) > \max\{I_{\lambda,\mu}(0), I_{\lambda,\mu}(v)\}$ for all $\mu \in [\delta, 1]$, where*

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = v\}.$$

- (c) *there exists $M_0 > 0$ independent of λ and μ such that $c_{\lambda,\mu} \leq M_0$*

Proof. (a) Ω is an open nonempty set in \mathbb{R}^3 by (V_3) , without loss of generality, we assume $0 \in \Omega$ and then there exists $\rho_0 > 0$ such that $B_{\rho_0}(0) \subset \Omega$. Let $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfy that $\text{supp } \psi \subset B_{\rho_0}(0)$ and $\psi_\theta = \theta^{s+t} \psi(\theta x)$, then $\text{supp } \psi_\theta \subset B_{\rho_0}(0)$ if $\theta > 1$. Hence for $\theta > 1$ and $V(x) \equiv 0$ in Ω , one has

$$0 \leq \int_{\mathbb{R}^3} V(x) \psi_\theta^2 dx = \int_{\text{supp } \psi} V(x) \psi_\theta^2 dx \leq \int_{B_{\rho_0}(0)} V(x) \psi_\theta^2 dx \leq \int_{\Omega} V(x) \psi_\theta^2 dx = 0.$$

In view of Lemma 2.2 (a) and (2.13), we have that

$$\begin{aligned} I_{\lambda,\delta}(\psi_\theta) &\leq \frac{\theta^{4s+2t-3}}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \psi|^2 dx + \theta^{4s+2t-3} |K|_\infty \int_{\mathbb{R}^3} \phi_\psi^t \psi^2 dx - \theta^{-3} \delta \int_{\mathbb{R}^3} F(\theta^{s+t} \psi) dx \\ &\leq \frac{\theta^{4s+2t-3}}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \psi|^2 dx + \theta^{4s+2t-3} |K|_\infty \int_{\mathbb{R}^3} \phi_\psi^t \psi^2 dx - \theta^{(s+t)\gamma-3} \delta \int_{\mathbb{R}^3} |\psi|^\gamma dx \\ &\rightarrow -\infty \end{aligned} \quad (2.17)$$

as $\theta \rightarrow +\infty$ because $\gamma > \frac{4s+2t}{s+t}$. Therefore we can take $v = \psi_{\theta_0}$ for some sufficiently large θ_0 , thus $I_{\lambda,\mu}(v) \leq I_{\lambda,\delta}(v) < 0$ for all $\mu \in [\delta, 1]$.

- (b) By means of (2.4) and (2.12), one has

$$I_{\lambda,\mu}(u) \geq \frac{1}{2} \|u\|_\lambda^2 - \epsilon \|u\|_\lambda^2 - C_\epsilon \|u\|_\lambda^q.$$

Let $\epsilon = \frac{1}{4}$, then $I_{\lambda,\mu}(u) > 0$ when $q > 2$ and $\|u\|_\lambda = \rho > 0$ small.

(c) let $\tilde{\gamma}(\theta) = v_\theta = \theta^{s+t}v(\theta x)$, where v is given by (a). Recalling the definition of e and Γ given by (b), one has $\tilde{\gamma} \in \Gamma$. Therefore we have that

$$c_\lambda \triangleq \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I_{\lambda,\mu}(\gamma(\theta)) \leq \max_{\theta \in [0,1]} I_{\lambda,\mu}(v_\theta) \leq \max_{\theta \geq 0} I_{\lambda,\delta}(\psi_\theta).$$

Using (2.17), $I_{\lambda,\delta}(\psi_\theta) \rightarrow -\infty$ as $\theta \rightarrow \infty$. Also we have $I_{\lambda,\delta}(\varphi_\theta) > 0$ for $\theta > 0$ small enough. Consequently, $c_\lambda \leq M_0 < +\infty$, where M_0 is independent on λ and μ . \square

3. THE PROOF OF THEOREM 1.1

In this section, we will prove the Theorem 1.1 in detail. Firstly we introduce the following Pohožaev identity (see [40]):

Lemma 3.1. (*Pohožaev identity*) Let $u \in H^s(\mathbb{R}^3)$ be a critical point of the functional $I_{\lambda,\mu}$ ($\forall \mu \in [\delta, 1]$) given by (2.16), then we have the following Pohožaev identity:

$$\begin{aligned} P_{\lambda,\mu}(u) \triangleq & \frac{3-2s}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} \lambda V(x) |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \lambda(x, \nabla V) |u|^2 dx \\ & + \frac{2t+3}{4} \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (x, \nabla K) \phi_u^t u^2 dx - 3\mu \int_{\mathbb{R}^3} F(u) dx \equiv 0. \end{aligned} \quad (3.1)$$

Lemma 3.2. Let $\{u_n\}$ be a bounded (PS) sequence of the functional $I_{\lambda,\mu}$ ($\forall \mu \in [\delta, 1]$) at the level $c > 0$, then for any $M > c$, there exists $\Lambda = \Lambda(M) > 0$ such that $\{u_n\}$ contains a strongly convergent subsequence in E_λ for all $\lambda > \Lambda$.

Proof. Since $\{u_n\}$ is bounded in E_λ , then there exists $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ , $u_n \rightarrow u$ in $L_{\text{loc}}^m(\mathbb{R}^3)$ with $m \in [1, 2_s^*)$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^3 . To show the proof clearly, we will split it into several steps:

Step 1: $I'_{\lambda,\mu}(u) = 0$ and $I_{\lambda,\mu}(u) \geq 0$.

To show $I'_\lambda(u) = 0$, since $C_0^\infty(\mathbb{R}^3)$ is dense in E_λ , then it suffices to show

$$\langle I'_{\lambda,\mu}(u), \varphi \rangle = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^3).$$

It is totally similar to the proof of [36, (3.2)] that

$$\int_{\mathbb{R}^3} f(u_n) \varphi dx \rightarrow \int_{\mathbb{R}^3} f(u) \varphi dx.$$

Using the above formula and (2.15), one has

$$\langle I'_{\lambda,\mu}(u), \varphi \rangle = \lim_{n \rightarrow \infty} \langle I'_{\lambda,\mu}(u_n), \varphi \rangle = 0.$$

Since u is a critical point of $I_{\lambda,\mu}$, then by (3.1) one has

$$I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u) - \frac{1}{(s+t)\gamma - 3} [(s+t) \langle I'_{\lambda,\mu}(u), u \rangle - P_{\lambda,\mu}(u)]$$

$$\begin{aligned}
&= \frac{(s+t)\gamma - (4s+2t)}{2[(s+t)\gamma - 3]} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{s+t}{(s+t)\gamma - 3} \int_{\mathbb{R}^3} [uf(u) - \gamma F(u)] dx \\
&\quad + \frac{(s+t)(\gamma - 2)}{2[(s+t)\gamma - 3]} \int_{\mathbb{R}^3} \lambda V(x) u^2 dx + \frac{1}{2[(s+t)\gamma - 3]} \int_{\mathbb{R}^3} \lambda(x, \nabla V) u^2 dx \\
&\quad + \frac{(s+t)\gamma - (4s+2t)}{4[(s+t)\gamma - 3]} \int_{\mathbb{R}^3} K(x) \phi_u^t u^2 dx + \frac{1}{2[(s+t)\gamma - 3]} \int_{\mathbb{R}^3} (x, \nabla K) \phi_u^t u^2 dx \\
&\geq \frac{(s+t)\gamma - (4s+2t)}{2[(s+t)\gamma - 3]} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \geq 0,
\end{aligned}$$

where we have used the fact $\gamma > \frac{4s+2t}{s+t}$ implies that $(s+t)\gamma > 3$.

Step 2: $u_n \rightarrow u$ in E_λ .

Let $v_n \triangleq u_n - u$, by (2.14), (2.15) and the Brézis-Lieb lemma [14], one has

$$I_{\lambda,\mu}(v_n) = I_{\lambda,\mu}(u_n) - I_{\lambda,\mu}(u) + o(1) \quad \text{and} \quad I'_\lambda(v_n) = I'_\lambda(u_n) + o(1). \quad (3.2)$$

As a consequence of the condition (V_2) and the locally compact Sobolev imbedding theorem, one has

$$\begin{aligned}
\int_{\mathbb{R}^3} v_n^2 dx &= \int_{\{V \geq c\}} v_n^2 dx + \int_{\{V < c\}} v_n^2 dx = \int_{\{V \geq c\}} v_n^2 dx + o(1) \\
&\leq \frac{1}{\lambda c} \int_{\{V \geq c\}} \lambda V(x) v_n^2 dx + o(1) \leq \frac{1}{\lambda c} \|v_n\|_\lambda^2 + o(1)
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{\mathbb{R}^3} |v_n|^q dx &= \left(\int_{\mathbb{R}^3} |v_n|^2 dx \right)^{\frac{2_s^*-q}{2_s^*-2}} \left(\int_{\mathbb{R}^3} |v_n|^{2_s^*} dx \right)^{\frac{q-2}{2_s^*-2}} \\
&\leq \left(\frac{1}{\lambda c} \right)^{\frac{2_s^*-q}{2_s^*-2}} \|v_n\|_\lambda^{2 \frac{2_s^*-q}{2_s^*-2}} S_s^{-\frac{2_s^*}{2} \frac{q-2}{2_s^*-2}} \|v_n\|_\lambda^{2_s^* \frac{q-2}{2_s^*-2}} + o(1) \\
&= \left(\frac{1}{\lambda c} \right)^{\frac{2_s^*-q}{2_s^*-2}} S_s^{-\frac{2_s^*}{2} \frac{q-2}{2_s^*-2}} \|v_n\|_\lambda^q + o(1)
\end{aligned} \quad (3.3)$$

Using $I_{\lambda,\mu}(u) \geq 0$ in Step 1, (f_3) and (3.2), we derive

$$\begin{aligned}
M_0 &\geq c - I_\lambda(u) = I_\lambda(v_n) - \frac{1}{\gamma} \langle I'_\lambda(v_n), v_n \rangle + o(1) \\
&= \frac{\gamma-2}{2\gamma} \|v_n\|_\lambda^2 + \frac{\gamma-4}{4\gamma} \int_{\mathbb{R}^3} K(x) \phi_{v_n}^t v_n^2 dx + \int_{\mathbb{R}^3} \left[\frac{1}{\gamma} v_n f(v_n) - F(v_n) \right] dx \\
&\stackrel{(2.14)}{\geq} \frac{\gamma-2}{2\gamma} \|v_n\|_\lambda^2 + o(1) \stackrel{(2.5)}{\geq} \frac{\gamma-2}{2\gamma} \left(2|\{V < c\}| \right)^{-\frac{2}{q} \frac{2_s^*-q}{2_s^*-2}} S_s \|v_n\|_q^2 + o(1)
\end{aligned} \quad (3.4)$$

when $\lambda \geq c^{-1} |\{V < c\}|^{-\frac{2_s^*-2}{2_s^*}} S_s$.

Combing with (3.3) and (3.4), for any $\lambda \geq c^{-1}|\{V < c\}|^{-\frac{2^*-2}{2s^*}} S_s$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |v_n|^q dx &\leq \left(\int_{\mathbb{R}^3} |v_n|^q dx \right)^{\frac{q-2}{q}} \left(\int_{\mathbb{R}^3} |v_n|^q dx \right)^{\frac{2}{q}} \\ &\leq \left(\frac{2M\gamma}{S_s(\gamma-2)} \right)^{\frac{q-2}{2}} \left(2|\{V < c\}| \right)^{\frac{q-2}{q} \frac{2^*-q}{2s^*}} \left(\frac{1}{\lambda c} \right)^{\frac{2}{q} \frac{2^*-q}{2s^*-2}} S_s^{-\frac{2^*}{q} \frac{q-2}{2s^*-2}} \|v_n\|_\lambda^2 \end{aligned}$$

which reveals that

$$\begin{aligned} o(1) &= \langle I'_\lambda(v_n), v_n \rangle \\ &= \|v_n\|_\lambda^2 + \int_{\mathbb{R}^3} K(x) \phi_{v_n}^t v_n^2 dx - \int_{\mathbb{R}^3} v_n f(v_n) dx \\ &\geq \|v_n\|_\lambda^2 - \epsilon \int_{\mathbb{R}^3} v_n^2 dx - C_\epsilon \int_{\mathbb{R}^3} |v_n|^q dx \\ &\stackrel{(2.5)}{\geq} \left\{ 1 - \epsilon \left(2|\{V < c\}| \right)^{\frac{2^*-2}{2s^*}} S_s^{-\frac{r}{2}} \right. \\ &\quad \left. - C_\epsilon \left(\frac{2M\gamma}{S_s(\gamma-2)} \right)^{\frac{q-2}{2}} \left(2|\{V < c\}| \right)^{\frac{q-2}{q} \frac{2^*-q}{2s^*}} \left(\frac{1}{\lambda c} \right)^{\frac{2}{q} \frac{2^*-q}{2s^*-2}} S_s^{-\frac{2^*}{q} \frac{q-2}{2s^*-2}} \right\} \|v_n\|_\lambda^2 \end{aligned}$$

when $\lambda \geq c^{-1}|\{V < c\}|^{-\frac{2^*-2}{2s^*}} S_s$. Therefore if we take $\epsilon > 0$ sufficiently small, then there exists $\Lambda = \Lambda(M) > c^{-1}|\{V < c\}|^{-\frac{2^*-2}{2s^*}} S_s$ such that $\|v_n\|_\lambda \rightarrow 0$ as $n \rightarrow \infty$. \square

As a direct consequence Proposition 2.4, Lemma 2.5 and Lemma 3.2, there exist two sequences $\{\mu_n\} \subset [\delta, 1]$ and $\{u_n\} \subset E_\lambda \setminus \{0\}$ (we denote $\{u(\mu_n)\}$ by $\{u_n\}$ just for simplicity) such that

$$I'_{\lambda, \mu_n}(u_n) = 0, \quad I_{\lambda, \mu_n}(u_n) = c_{\lambda, \mu_n} \quad \text{and} \quad \mu_n \rightarrow 1^-. \quad (3.5)$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We first claim that the sequence given by (3.5) is bounded. In fact, recalling (c) of Lemma 2.5, (3.1), the assumptions (V_4) and (K) , one has

$$\begin{aligned} M_0 &\geq c_{\lambda, \mu_n} = I_{\lambda, \mu_n}(u_n) - \frac{1}{(s+t)\gamma-3} [(s+t)\langle I'_{\lambda, \mu_n}(u_n), u_n \rangle - P_{\lambda, \mu_n}(u_n)] \\ &= \frac{(s+t)\gamma - (4s+2t)}{2[(s+t)\gamma-3]} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{s+t}{(s+t)\gamma-3} \int_{\mathbb{R}^3} [u_n f(u_n) - \gamma F(u_n)] dx \\ &\quad + \frac{(s+t)(\gamma-2)}{2[(s+t)\gamma-3]} \int_{\mathbb{R}^3} \lambda V(x) u_n^2 dx + \frac{1}{2[(s+t)\gamma-3]} \int_{\mathbb{R}^3} \lambda(x, \nabla V) u_n^2 dx \\ &\quad + \frac{(s+t)\gamma - (4s+2t)}{4[(s+t)\gamma-3]} \int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx + \frac{1}{2[(s+t)\gamma-3]} \int_{\mathbb{R}^3} (x, \nabla K) \phi_{u_n}^t u_n^2 dx \\ &\geq \frac{(s+t)\gamma - (4s+2t)}{2[(s+t)\gamma-3]} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \end{aligned} \quad (3.6)$$

which shows that $|(-\Delta)^{\frac{s}{2}} u_n|_2$ is bounded. By interpolation inequality, for $q \in (2, 2_s^*)$ one has

$$\begin{aligned} |u_n|_q^q &\leq |u_n|_2^{2\xi} |u_n|_{2_s^*}^{2_s^*(1-\xi)} \stackrel{(2.5)}{\leq} C \|u\|_\lambda^{2\xi} |u_n|_{2_s^*}^{2(1-\xi)} \\ &\stackrel{(2.3)}{\leq} C \|u_n\|_\lambda^{2\xi} S_s^{-(1-\xi)} |(-\Delta)^{\frac{s}{2}} u_n|_2^{1-\xi} \leq C \|u_n\|_\lambda^{2\xi}, \end{aligned} \quad (3.7)$$

where $\xi = \frac{2_s^*-q}{2_s^*-2} \in (0, 1)$. Therefore by (2.13), one has

$$\begin{aligned} M_0 &\geq c_{\lambda, \mu_n} = I_{\lambda, \mu_n}(u_n) \geq \frac{1}{2} \|u_n\|_\lambda^2 - \epsilon \|u_n\|_\lambda^2 - C_\epsilon |u_n|_q^q \\ &\geq \frac{1}{4} \|u_n\|_\lambda^2 - C \|u_n\|_\lambda^{2\xi}, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E_λ because $\xi \in (0, 1)$.

Since $\mu_n \rightarrow 1^-$, we claim that $\{u_n\}$ is a $(PS)_{c_{\lambda,1}}$ sequence of the functional $I_\lambda = I_{\lambda,1}$. In fact, as a consequence of Lemma 2.4 (c) we obtain that

$$\lim_{n \rightarrow \infty} I_{\lambda,1}(u_n) = \left(\lim_{n \rightarrow \infty} I_{\lambda, \mu_n}(u_n) + (\mu_n - 1) \int_{\mathbb{R}^3} F(u_n) dx \right) \stackrel{(2.12)}{=} \lim_{n \rightarrow \infty} c_{\lambda, \mu_n} = c_{\lambda,1}$$

and for all $\psi \in H^s(\mathbb{R}^3) \setminus \{0\}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|\langle I'_{\lambda,1}(u_n), \psi \rangle|}{\|\psi\|} &= \lim_{n \rightarrow \infty} \frac{|\langle I'_{\lambda, \mu_n}(u_n), \psi \rangle + (\mu_n - 1) \int_{\mathbb{R}^3} f(u_n) \psi dx|}{\|\psi\|} \\ &= \lim_{n \rightarrow \infty} \frac{|\mu_n - 1| \left| \int_{\mathbb{R}^3} f(u_n) \psi dx \right|}{\|\psi\|} \\ &\stackrel{(2.12)}{\leq} \lim_{n \rightarrow \infty} |\mu_n - 1| (\epsilon \|u_n\| + C_\epsilon \|u_n\|^{q-1}) \rightarrow 0, \end{aligned}$$

which imply that $\{u_n\}$ is a $(PS)_{c_{\lambda,1}}$ sequence of $I_\lambda = I_{\lambda,1}$ at the level $c_{\lambda,1} > 0$, where we have used the fact that $\{u_n\}$ is bounded in E . Consequently by Lemma 3.2, there exists a subsequence still denoted by itself such that $u_n \rightarrow u$ in E which implies that $I_\lambda(u) = c_{\lambda,1} > 0$ and $I'_\lambda(u) = 0$.

Inspired by J. Sun and S. Ma [38], to obtain a ground state solution we set

$$m = \inf \{I_\lambda(u) : u \in E \setminus \{0\}, I'_\lambda(u) = 0\}.$$

We claim that $m > 0$. Indeed, similar to the Step 1 in the proof of Lemma 3.2, one has $m \geq 0$. In order to show $m > 0$, we suppose that $m = 0$. Take a minimizing sequence $\{w_n\}$ such that $I'_\lambda(w_n) = 0$ and $I_\lambda(w_n) \rightarrow 0$. Using $I'_\lambda(w_n) = 0$ and (2.12), one has

$$\|w_n\|^2 \leq \|w_n\|_\lambda^2 \leq \int_{\mathbb{R}^3} f(w_n) w_n \leq \frac{1}{2} \|w_n\|^2 + C |w_n|_q^q \stackrel{(2.4)}{\leq} \frac{1}{2} \|w_n\|^2 + C \|w_n\|^q \quad (3.8)$$

which implies that $\|w_n\| \geq C > 0$ for some C independent of n . On the other hand, Using $I_\lambda(w_n) \rightarrow 0$ and $I'_\lambda(w_n) = 0$, as (3.6) we have $|(-\Delta)^{\frac{s}{2}} w_n|_2 \rightarrow 0$. Similar to the Step 1 in the proof of Lemma 3.2, $\{w_n\}$ is bounded in E_λ . Hence $|w_n|_q \rightarrow 0$ by (3.7). Using (3.8) again, we have $0 \leq \|w_n\|^2 \leq C |w_n|_q^q \rightarrow 0$, which is a contradiction!

Suppose that there exists a sequence $\{u_n\} \subset E \setminus \{0\}$ such that $I'_\lambda(u_n) = 0$ and $I_\lambda(u_n) \rightarrow m$. We can conclude that $\{u_n\}$ is bounded in E , and then $\{u_n\}$ is (PS) sequence at the level $m > 0$. By Lemma 3.2, passing to a subsequence if necessary, $u_n \rightarrow u$ in E_λ . Hence we have that $I_\lambda(u) = m > 0$ and $I'_\lambda(u) = 0$ which shows that u is a nontrivial critical point of I_λ given by (2.11). It follows from [41, Proposition 4.4] that u is positive. Therefore (u, ϕ_u) is a positive ground state to system (1.1). The proof is complete. \square

4. CONCENTRATION FOR THE NONTRIVIAL SOLUTIONS OBTAINED IN THEOREM 1.1

Before we study the concentration results, let us recall the Vanishing lemma for fractional Sobolev space as follows.

Lemma 4.1. (see [34, Lemma 2.4]) *Assume that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$ for $\alpha \in (0, 1)$ and satisfies*

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_\rho(y)} |u_n|^2 dx = 0,$$

for some $\rho > 0$. Then $u_n \rightarrow 0$ in $L^m(\mathbb{R}^3)$ for every $2 < m < 2_\alpha^*$.

We adapt the idea used in [9, 47] to prove Theorem 1.4.

Proof of Theorem 1.4. For any sequence $\lambda_n \rightarrow \infty$, we denote $\{u_n\}$ to be the positive ground state solutions $\{u_{\lambda_n}\}$ obtained in Theorem 1.1. It is similar to the proof in Theorem 1.1 that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and going to a subsequence if necessary we can assume that $u_n \rightarrow u_0$ in E , $u_n \rightarrow u_0$ in $L_{\text{loc}}^p(\mathbb{R}^3)$ with $p \in [1, 2_s^*)$ and $u_n \rightarrow u_0$ in *a.e.* in \mathbb{R}^3 . Using Fatou's lemma, one has

$$0 \leq \int_{\mathbb{R}^3 \setminus V^{-1}(0)} V(x) u_0^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(x) u_n^2 dx \leq \liminf_{n \rightarrow \infty} \frac{1}{\lambda_n} \|u_n\|_{\lambda_n}^2 \leq \lim_{n \rightarrow \infty} \frac{C}{\lambda_n} = 0$$

which implies that $u_0 = 0$ *a.e.* in $\mathbb{R}^3 \setminus V^{-1}(0)$, then we have that $u_0 \in H_0^s(\Omega)$ because $\Omega = \text{int} V^{-1}(0)$ by (V_3) . Now for any $\varphi \in C_0^\infty(\Omega)$, and since $\langle I'_{\lambda_n}(u_n), \varphi \rangle = 0$, we can easily check that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_0 (-\Delta)^{\frac{s}{2}} \varphi + \int_{\mathbb{R}^3} K(x) \phi_{u_0}^t u_0 \varphi dx - \int_{\mathbb{R}^3} f(u_0) \varphi dx = 0.$$

As $C_0^\infty(\Omega)$ is dense in $H_0^s(\Omega)$, u_0 is a solution of (1.5).

We claim that $u_n \rightarrow u_0$ in $L^q(\mathbb{R}^3)$ for $q \in (2, 2_s^*)$. Arguing it by indirectly, then by Lemma 4.1 there exists $\{y_n\} \subset \mathbb{R}^3$, $\rho > 0$ and $\delta_0 > 0$ such that

$$\int_{B_\rho(y_n)} (u_n - u_0)^2 dx \geq \delta_0 > 0,$$

where $|y_n| \rightarrow \infty$ which implies that $|B_\rho(y_n) \cap \{V < c\}| \rightarrow 0$. By Hölder's inequality

$$\int_{B_\rho(y_n) \cap \{V < c\}} (u_n - u_0)^2 dx \rightarrow 0$$

which implies that for sufficiently large n one has

$$\int_{B_\rho(y_n) \cap \{V < c\}} (u_n - u_0)^2 dx \leq \frac{\delta_0}{2}.$$

Therefore for sufficiently large n and $u_0 = 0$ a.e. in $\mathbb{R}^3 \setminus V^{-1}(0)$ give that

$$\begin{aligned} \|u_n\|_{\lambda_n}^2 &\geq \lambda_n \int_{\mathbb{R}^3} V(x) u_n^2 dx \geq \lambda_n c \int_{B_\rho(y_n) \cap \{v \geq c\}} u_n^2 dx = \lambda_n c \int_{B_\rho(y_n) \cap \{v \geq c\}} (u_n - u_0)^2 dx \\ &= \lambda_n c \left(\int_{B_\rho(y_n)} (u_n - u_0)^2 dx - \int_{B_\rho(y_n) \cap \{v < c\}} (u_n - u_0)^2 dx \right) \geq \frac{\delta_0}{2} \lambda_n c \rightarrow \infty \end{aligned}$$

which yields a contradiction! Hence $u_n \rightarrow u_0$ in $L^q(\mathbb{R}^3)$ for $q \in (2, 2_s^*)$ which implies that

$$\int_{\mathbb{R}^3} f(u_n) u_n dx - \int_{\mathbb{R}^3} f(u_n) u_0 dx = o(1)$$

by the Strass compactness theorem in [12].

We now show that $u_n \rightarrow u_0$ in E . In fact, by $\langle I'_{\lambda_n} u_n, u_n \rangle = \langle I'_{\lambda_n} u_n, u_0 \rangle = 0$

$$\|u_n\|_{\lambda_n}^2 + \int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx = \int_{\mathbb{R}^3} f(u_n) u_n dx \quad (4.1)$$

and

$$(u_n, u_0)_{\lambda_n} + \int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n u_0 dx = \int_{\mathbb{R}^3} f(u_n) u_0 dx.$$

In view of the definition A_2 in the proof of (2.14), one has

$$\int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} K(x) \phi_{u_n}^t u_n u_0 dx = o(1).$$

Hence by the above four formulas we have that

$$\lim_{n \rightarrow \infty} \|u_n\|_{\lambda_n}^2 = \lim_{n \rightarrow \infty} (u_n, u_0)_{\lambda_n} = \lim_{n \rightarrow \infty} (u_n, u_0) = \|u_0\|^2.$$

Also since the norm is lower semicontinuous, then

$$\|u_0\|^2 \leq \inf_{n \rightarrow \infty} \lim \|u_n\|^2 \leq \inf_{n \rightarrow \infty} \lim \|u_n\|_{\lambda_n}^2,$$

and thus $u_n \rightarrow u_0$ in E .

Finally, we show $u_0 \neq 0$. Using (2.3), (2.12) and (4.1), we drive

$$\|u_n\|^2 \leq \|u_n\|_{\lambda_n}^2 \leq \int_{\mathbb{R}^3} f(u_n) u_n dx \leq \frac{1}{2} \|u_n\|^2 + C \|u_n\|_q^q$$

which implies $\|u_0\|^2 \leq C \|u_0\|^q$ together with $u_n \rightarrow u_0$ in E . Therefore $u_0 \neq 0$ for $q \in (2, 2_s^*)$. The proof is complete. \square

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